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Note

Notes on Approximation

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1. WHAT ARE THE SCHOENBERG AND BIRKHOFF SPACES OF SPLINES?

These spaces of splines are *smoothness spaces*, that is, they require some measure of smoothness for their elements S. They are usually defined by postulating certain smoothness properties of S at their knots (= breakpoints). But is it not possible to define the spaces "axiomatically," by means of some intrinsic properties, without mentioning the knots of each S? This is possible, and not difficult. To be specific, we do this for splines on $\mathbb{R}$.

An analogy is to rearrangement–invariant spaces X (of measurable functions on $\mathbb{R}$). They may be described by assuming that if $|f| \prec |g|$ (this is the Hardy–Littlewood–Pólya quasi-order relation) and if $g \in X$, then also $f \in X$.

A function S on $\mathbb{R}$ is a spline of order $r = 1, 2, ...$ if there is a finite or infinite increasing sequence of points $T := (t_i)$, $i \in \mathbb{Z}$, on $\mathbb{R}$, $|t_i| \to \infty$ if $i \to \pm\infty$, so that on each interval (t_i, t_{i+1}), and on the intervals $(-\infty, t_1)$ or (t_p, ∞) (if there is a first or a last t_i) it is a polynomial of degree $r-1$, and of degree exactly $r-1$ on one of the intervals. At the t_i, S and its derivatives (which are also splines) are defined by continuity, if possible; otherwise they are not defined. A point t_i is a breakpoint of S, if one of the derivatives $S^{(m)}$ is not defined at t_i.

Let $\mathcal{S}_r := \mathcal{S}_r(\mathbb{R})$ be the linear space of all splines of order $\leqslant r$ on $\mathbb{R}$.

A Schoenberg space $\mathcal{S}_r(T, \mathbf{m}, \mathbb{R})$ is defined (see [1]) by $r = 1, 2, ...$, a sequence $T = (t_i)$ and by a corresponding sequence $\mathbf{m} = (m_i)$ of integers, $0 \leqslant m_i < r$. A spline S belongs to this space if it is of order $\leqslant r$, if its breakpoints are among the t_i, and if all derivatives $S^{(m)}$, $m < m_i$, exist at t_i.

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For the Birkhoff space of splines, instead of the m_i, for each i we have integers $0 \leqslant m_{i,1} < \cdots < m_{i,p} < r$, $p = p(i)$; then S belongs to the space if all $S^{(m)}$ exist at t_i, except perhaps those with $m = m_{i,j}$.

For two splines S, $S_1 \in \mathcal{S}_r$ (with breakpoints that are not specified), we define two smoothness relations. We say that S is $(*)$ at least as smooth as S_1 if S has all derivatives of orders $\leqslant m$ at $x \in \mathbb{R}$ whenever S_1 has these derivatives at x. We say that S is $(**)$ at least as smooth as S_1 if S has the derivative of order m at $x \in \mathbb{R}$ whenever S_1 has this derivative at x.

A subspace $\mathcal{S}$ of $\mathcal{S}_r$ has the smoothness property $(*)$ (or $(**)$) if $S_1 \in \mathcal{S}$ implies $S \in \mathcal{S}$ for each S that is $(*)$ (or $(**)$) at least as smooth as S_1.

THEOREM 1. *A subspace $\mathcal{S}$ of $\mathcal{S}_r$ is a Schoenberg* (*or a Birkhoff*) *spline space if and only if it is* $(*)$ (*or* $(**)$) *smooth and is closed with respect to the uniform convergence on compact sets.*

Proof. We have to prove only the sufficiency of the conditions. Let $\mathcal{S}$ be $(*)$- or $(**)$-smooth. Let (t_i) be an increasing sequence in $\mathbb{R}$, each t_i being a breakpoint of some spline $S \in \mathcal{S}$. We prove that $|t_i| \to \infty$ if $i \to \infty$ or $i \to -\infty$. Let

$$\phi_m(x, a) = \begin{cases} (x-a)_+^m, & a \geqslant 0 \\ (x-a)_-^m, & a < 0. \end{cases} \tag{1.1}$$

From the smoothness property, for each t_i there is an m_i, $0 \leqslant m_i < r$, so that $\phi_i(x) := \phi_{m_i}(x; t_i)$ belongs to $\mathcal{S}$. Then also $f = \sum (|j|+1)^{-2} \phi_j \in \mathcal{S}$, a contradiction if $|t_i| \to \infty$ is not satisfied, since then $f \notin \mathcal{S}_r$. We can now combine all breakpoints of all $S \in \mathcal{S}$ into an increasing sequence $T := (t_i)$ (with $|t_i| \to \infty$ for $i \to \pm\infty$ in the infinite case).

If $\mathcal{S}$ is $(*)$-*smooth*, for each i we let m be the smallest integer so that one of the splines $S \in \mathcal{S}$ does not have a derivative $S^{(m_i)}(t_i)$. Let $\phi_i(x) = \phi_{m_i}(x, t_i)$. Then the function

$$S_0 = \sum \phi_i \tag{1.2}$$

belongs to $\mathcal{S}$, since on each interval $[-A, A]$, all terms of the sum with sufficiently large i are zero. This S_0 is a universal ("the worst") spline of $\mathcal{S}$: A spline $S \in \mathcal{S}_r$ belongs to $\mathcal{S}$ if and only if S is $(*)$ at least as smooth as S_0. This proves that $\mathcal{S}$ is the Schoenberg space $\mathcal{S}_r(T, \mathbf{m}, \mathbb{R})$.

If $\mathcal{S}$ is $(**)$-*smooth*, the proof is the same except that for each t_i we let the $m_{i,j}$, $0 \leqslant m_{i,1} < \cdots < m_{i,p} < r$, be all m for which the derivative $S^{(m)}(t_i)$ does not exist at t_i for some $S \in \mathcal{S}$. The universal spline S_0 is then given by

$$S_0(x) = \sum_{i,j} \phi_{m_{i,j}}(x, t_i). \quad \blacksquare \tag{1.3}$$

2. A Bernstein-Type Inequality for Exponential Sums

If X_n is a linear n-dimensional space of differentiable functions on $[a, b]$, imbedded in a normed space X, then sometimes we have

$$\|f'\|_X \leqslant C(n) \|f\|_X. \tag{2.1}$$

We hope to determine the best factor $C(n)$ exactly, or asymptotically. In more complicated cases, X_n is not linear, and the correct inequality has $\|f'\|$ in (2.1) with norm in a *different space* than X. Thus, Dolž'enko ineqality [2] for rational functions R_n of degree $\leqslant n$ on $[0, 1]$ has the form

$$\|R_n'\|_1 \leqslant 2n \|R_n\|_\infty. \tag{2.2}$$

Inequalities of Pekarski [3] for the derivatives of higher order of R_n are also of this type. Here we develop, using an argument of E. Schmidt [4], a Bernstein-type inequality for the *extended exponential sums* on $[a, b]$,

$$g(x) = \sum_{\nu=1}^{p} P_\nu(x) e^{\lambda_\nu x}, \qquad \sum_{\nu=1}^{p} (\partial P_\nu + 1) \leqslant n, \tag{2.3}$$

where ∂P_ν is the degree of the polynomial P_ν. They form the family $\mathscr{E}_n$, where the coefficients of the polynomials P_ν and the λ_ν are free real parameters. For fixed λ_ν and fixed $m_\nu = \partial P_\nu$, the $g \in \mathscr{E}_n$ are spanned by the elements of the Haar system

$$e^{\lambda_1 x}, \; x e^{\lambda_1 x}, \ldots, x^{m_p} e^{\lambda_p x}. \tag{2.4}$$

It follows that a non-trivial function g cannot have more than $n-1$ zeros. The same applies to their derivatives, since $g^{(k)} \in \mathscr{E}_n$, $k = 1, 2, \ldots$.

The family $\mathscr{E}_n[a, b]$ contains *zoom functions*, that is, strictly increasing functions g satisfying

$$\begin{aligned} 0 \leqslant g(x) \leqslant \varepsilon, \qquad & a \leqslant x \leqslant b - \varepsilon \\ g(b) \geqslant A & \end{aligned} \tag{2.5}$$

for arbitrary $\varepsilon > 0$, $A > 0$. An example is given by $g(x) = \varepsilon e^{\lambda(x + \varepsilon - b)}$, where λ is sufficiently large.

This shows that for $\mathscr{E}_n$ there do not exist exact parallels to the Bernstein inequality for polynomials. Inequalities that are true are weaker.

In the following theorem the norm is of the same type on both sides of the inequality, but for the derivative it is computed on a smaller interval. We improve the result of Schmidt somewhat; he gives an unspecified constant $C(n)$ in (2.6) instead of $C \exp(\alpha \log^2 n)$.

THEOREM 2. *For each* $\alpha > \frac{1}{2}$, *there is a constant* $C > 0$ *so that for all* $g \in \mathscr{E}_n[a, b]$ *and for* $\delta < \frac{1}{2}(b-a)$,

$$\|g'\|_\infty [a+\delta, b-\delta] \leqslant C\delta^{-1} n^{\alpha \log n} \|g\|_\infty [a, b]. \tag{2.6}$$

We need some facts about a function $f \in C^{n+1}[-1, 1]$ with $\|f\|_\infty \leqslant 1$.

1. In each interval $(a, b) \subset [-1, 1]$, $b - a = d$, there is a point y with

$$|f^{(k)}(y)| \leqslant r_k(d), \qquad r_k(d) := \left(\frac{2k}{d}\right)^k. \tag{2.7}$$

Indeed, $|\Delta_h^k f(x)| \leqslant 2^k \|f\| \leqslant 2^k$ if this difference is defined. This is the case if $h = d/k$, $x = a$. For some y, $\Delta_h^k f(x) = h^k f^{(k)}(y)$, and we get (2.7).

2. Let $[c, c+d]$ and $[c-d, c]$ be subsets of $[-1, 1]$, and let $f^{(k)}(c) \geqslant C$. Then for some c_1, c_1',

$$-f^{(k+1)}(c_1) \geqslant (C - r(d))\frac{1}{d}, \qquad c \leqslant c_1 \leqslant c+d, \tag{2.8}$$

$$f^{(k+1)}(c_1') \geqslant (C - r(d))\frac{1}{d}, \qquad c-d \leqslant c_1' \leqslant c. \tag{2.9}$$

Let $y \in (c, c+d)$ satisfy (2.7). We obtain (2.8) from

$$-\frac{1}{y-c}\{f^{(k)}(y) - f^{(k)}(c)\} \geqslant (C - r(d))\frac{1}{d}$$

and the mean value theorem. Inequality (2.9) is similar.

3. The following serves as a supplement to Rolle's theorem to obtain additional zeros of $f^{(k+1)}$.

Let $c < c_1$, $f^{(k)}(c) > 0$, $f^{(k+1)}(c_1) < 0$. If $f^{(k)}$ has zeros to the left of c, then for some y, $c \leqslant y < c_1$, $f^{(k+1)}(y) = 0$.

Indeed, if $x < c$ is the largest of the zeros, then necessarily $f^{(k+1)}(x) \geqslant 0$, and the conclusion follows.

Similarly, we have $f^{(k+1)}(y') = 0$ for some y', $c_1' < y' \leqslant c'$, if $f^{(k)}(c') > 0$, $f^{(k+1)}(c_1') > 0$, and if there are zeros of $f^{(k)}$ to the right of c'.

Proof of Theorem 2. Clearly, it is sufficient to prove the following. There exists an absolute constant $C > 0$ with the property that $g \in \mathscr{E}_n[-\delta, \delta]$ and $\|g\|_\infty [-\delta, \delta] \leqslant 1$ imply

$$g'(0) \leqslant C\frac{1}{\delta} n^{\alpha \log n}. \tag{2.10}$$

This will be achieved by assuming that

$$g'(0) \geqslant C_1 \tag{2.11}$$

and by obtaining a contradiction for some $C_1 < C\delta^{-1} n^{\alpha \log n}$. All derivatives of g belong to $\mathscr{E}_n$, but we can assume that $g \notin \mathscr{P}_n$.

We put $d_k = \delta n^{-1} \exp(\alpha_1 \log^2 k - \alpha_1 \log^2 n)$, $\frac{1}{2} < \alpha_1 < \alpha$, $k = 1, 2, ..., n$. Then

$$d_1 + d_2 + \cdots + d_n \leqslant \delta. \tag{2.12}$$

Also let $C_{k+1} = (C_k - r_k(d_k))\, d_k^{-1}$, $k = 2, ..., n$, and

$$C_1 = r_1(d_1) + d_1 r_2(d_2) + \cdots + d_1 \cdots d_{n-1} r_n(d_n); \tag{2.13}$$

then $C_k \geqslant 0$, $k = 1, ..., n+1$. For large k we have

$$\sum_{j=1}^{k-1} j \log^2 j \leqslant k \log^2 k - 2k \log k + \mathscr{O}(k);$$

hence for a properly chosen C

$$\begin{aligned} C_1 &= \sum_{k=1}^{n} (2k)^k \frac{d_1 \cdots d_{k-1}}{d_k^k} \\ &= \frac{n}{\delta} n^{\alpha_1 \log n} \sum_{k=1}^{n} (2k)^k \exp\{-2\alpha_1 k \log k + \mathscr{O}(k)\} \\ &\leqslant \frac{1}{\delta} n^{\alpha_1 \log n + 1} \sum_{k=1}^{\infty} \exp\{-(2\alpha_1 - 1) k \log k + \mathscr{O}(k)\} \\ &\leqslant \frac{C}{\delta} n^{\alpha \log n}. \end{aligned}$$

Using (2.8) and (2.9) with $C = C_1$, $c = 0$, $d = d_1$, $k = 1$ we get points c_2', c_2 for which $-d_1 \leqslant c_2' < 0 < c_2 \leqslant d_1$ and

$$-g''(c_2) \geqslant C_2, \qquad g''(c_2') \geqslant C_2. \tag{2.14}$$

At the kth step, we find c_{k+1}', c_{k+1} for which $c_k \leqslant c_{k+1} < c_k + d_k$, $c_k' - d_k \leqslant c_{k+1}' \leqslant c_k'$ and

$$(-1)^k g^{(k+1)}(c_{k+1}) \geqslant C_{k+1}, \qquad g^{(k+1)}(c_{k+1}') \geqslant C_{k+1}. \tag{2.15}$$

Clearly, $c_k, c_k' \in (-\delta, \delta)$, $k = 1, ..., n+1$.

We now prove tht for $k = 2, ..., n$, $g^{(k)}$ has at least $k-1$ zeros in (c_k', c_k). For $k = 2$ this follows from (2.14). Let this statement be true for some k.

Applying 3, we obtain two zeros of $g^{(k+1)}$ with $c_k \leqslant y < c_{k+1}$, $c'_{k+1} < y' \leqslant c'_k$. In (c'_k, c_k) this function has at least $k-2$ zeros by Rolle's theorem. This yields k zeros of $g^{(k+1)}$ in (c'_{k+1}, c_{k+1}).

For $g^{(n+1)}$ we get n zeros in $[-\delta, \delta]$, hence $g^{(n+1)} \equiv 0$, or $g \in \mathscr{P}_n$, a contradiction. ∎

Note added in proof. Y. Xu remarks that one can improve (2.6) somewhat, replacing $n^{\alpha \log n}$ by $n^{(1/2)\log n + 5}$. One takes

$$d_n = \frac{\delta}{n} \exp(-\alpha_k), \qquad \alpha_k = \sum_{j=k}^{n} \frac{\log(1+j) + A}{1+j}$$

with $A = 1 + \log 2$. This leads to the estimate $C_1 \leqslant (n^2/\delta)\, e^{-\alpha_1}$.

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